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On a difficulty in the Theory of the Attraction of Spheroids.

By James M'Cullagh, A.B.

Read May 28, 1832.

An approximate theorem, discovered by Laplace, and relating to the attraction of a solid slightly differing from a sphere, on a point placed at its surface, has given rise to many disputes among mathematicians.* I hope the question will be set in a clear light by the following remarks.

Let us consider the function which expresses the sum of every element of a solid divided by its distance from a fixed point, and let us denote it, as Laplace has done, by the letter V. It is necessary to find the value of V for a pyramid of indefinitely small angle, the fixed point being at its vertex. Calling ϕ the small solid angle of the pyramid (or the area which it intercepts on the surface of a sphere whose radius is unity and centre at the vertex), it is manifest that the element of the pyramid at the distance r from the vertex is $\phi r^2 dr$; dividing therefore by r, and integrating, we have $\frac{1}{2}\phi r^2$, or ϕ multiplied into half the square of the length, for the value of V.

Again, supposing the force to vary inversely as the square of the distance—the only hypothesis that can be of use in the present inquiry—the attraction of the same pyramid on a point at its vertex, and in the direction of its length, is manifestly equal to ϕr.

Let us now consider a solid of any shape, regular or irregular, terminated at one end by a plane to which the straight line PQ (*Fig.* 1,) is perpendicular at the point P; and let there be a sphere of any magnitude, whose diameter $P'Q$ is parallel to PQ. Let P'' be a fixed point, and from the points P, P', P'', draw three parallel straight lines Pp, $P'p'$, $P''p''$, the first two terminated by the surfaces of the solid and of the sphere, the third $P''p''$ in the same direction with them and equal to their difference, without regarding which of them is the greater, and suppose all the points p'', taken according to the same law, to trace the surface of a third solid. Let Pp, Pp', $P''p''$, be edges of three small pyramids with their other edges proceeding from $\mathbf{P}$, $\mathbf{P}'$, $\mathbf{P}''$, parallel, and having of course the same solid angle which we shall call ϕ, denoting by r, r', r'', their respective lengths, and by V, V', V'', the values of the function V for each of them. Drawing pR perpendicular to $\mathbf{P}\,Q$, the attraction of the pyramid $\mathbf{P}p$ in the direction of $\mathbf{P}\,Q$ will be equal to $\phi \times \mathbf{P}R$; call this attraction A, and let a be the radius of the sphere.

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* See Pontécoulant, *Théorie analytique du système du monde,* Tome II. p. 380; with the references there given.

Since r'' is the difference of r and r', we have $r^2+r'^2-r''^2=2\,rr'=2\,\mathbf{P}R\times\mathbf{P}'\,Q'$, and multiplying by $\frac{1}{2}\,\phi$ we find $\frac{1}{2}\,\phi r^2+\frac{1}{2}\,\phi r'^2-\frac{1}{2}\,\phi r''^2=2a\phi\times\mathbf{P}R$, that is $V+V'-V''=2aA$. The same thing is true for any other three pyramids similarly related to each other, throughout the whole extent of the three solids which are exhausted by them at the same time; and hence, if we now denote by V, V', V'', the *whole* values of the function V for the three solids, and by A the *whole* attraction of the first of them parallel to $\mathbf{P}\,Q$ on a point at $\mathbf{P}$, we shall still have $V+V'-V''=2aA$.

To express this general theorem in the notation of LAPLACE, we have merely to observe that the attraction A is synonymous with $-\left(\frac{dV}{dr}\right)$, and that the quantity V' for the sphere is equal to $\frac{4}{3}\pi a^2$. Substituting these values, we find

$$V+2a\left(\frac{dV}{dr}\right)=-\tfrac{4}{3}\pi a^2+V'';$$

an exact equation, differing from the approximate one of LAPLACE only in containing the quantity V'', and totally independent of the nature of the surface or of the magnitude of the sphere; the only things supposed being that all the lines drawn from $\mathbf{P}$ meet the surface again but once, and that no part of it passes beyond a plane through $\mathbf{P}$ at right angles to $\mathbf{P}\,Q$.

With respect to the limit of the quantity V'', it is obvious that if a hemisphere be described from $\mathbf{P}''$ as a centre with a radius equal to the greatest difference δ between the lines Pp, $P'p'$, the solid $\mathbf{P}''p''$ will lie wholly within this hemisphere, and consequently V'' will be less than the value of V for the hemisphere, that is, less than $\pi\delta^2$; for here all the little pyramids from the centre have the same length δ, and their bases are spread over the hemispherical surface; wherefore $V''=2\pi\times\frac{1}{2}\delta^2=\pi\delta^2$. All this is independent of any thing but the suppositions just mentioned.

If now $\mathbf{P}\,Q$ be supposed to be a spheroid of any sort, slightly differing from the sphere $\mathbf{P}'\,Q'$, and such that the line $\mathbf{P}\,Q$, perpendicular to the surface at $\mathbf{P}$, passes nearly through the centre, than all the differences, of which δ is the greatest, being of the first order, the quantity V'', which is less than $\pi\delta^2$, will be of the second order; and therefore neglecting, as LAPLACE has done, the quantities of that order, we get the theorem in question.

It may be well to apply the general theorem to the simple case in which the first solid is a sphere of the radius a', because both LAGRANGE and IVORY have used this case to show that the reasonings of LAPLACE are incorrect. In this instance, then, the surface described by the point p'' is that of a sphere whose radius is the difference between a and a'; and the values of V, V', V'', and A, are $\frac{4}{3}\pi a'^2$, $\frac{4}{3}\pi a^2$, $\frac{4}{3}\pi (a'-a)^2$ and $\frac{4}{3}\pi a'$, respectively.

Substituting these values in the equation $V+V'-V''=2aA$, and omitting the common factor $\frac{4}{3}\pi$, the resulting equation

$$a'^2+a^2-(a'-a)^2=2aa'$$

ought to be identical;—and so it manifestly is.

November, 1831.